Math 150, *Lecture Notes- Bonds* **Name____________**

Section 4.5 Integration by Substitution

Pattern Recognition

In this section you will study techniques for integrating composite functions. The discussion is split into two parts—*pattern recognition* and *change of variables*. Both techniques involve a ***u*-substitution.** With pattern recognition you perform the substitution mentally, and with change of variables you write the substitution steps.

The role of substitution in integration is comparable to the role of the Chain Rule in differentiation. Recall that for differentiable functions given by $y = F(u)$ and $u = g(x)$, the Chain Rule states that

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x).$$

From the definition of an antiderivative, it follows that

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

These results are summarized in the following theorem.

THEOREM 4.12 Antidifferentiation of a Composite Function

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

If $u = g(x)$, then $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C.$$

Examples 1 and 2 show how to apply Theorem 4.13 *directly*, by recognizing the presence of $f(g(x))$ and $g'(x)$. Note that the composite function in the integrand has an *outside function* f and an *inside function* g. Moreover, the derivative $g'(x)$ is present as a factor of the integrand.

Outside function

$$\int f(g(x))\underbrace{g'(x)}\,dx = F(g(x)) + C$$

Inside function

Derivative of inside function

Ex.1 Recognizing the $f\left(g(x)\right)g'(x)$ Pattern

Find $\int (x^2 - 9)^3(2x)\, dx$

Ex.2 Recognizing the $f\left(g(x)\right)g'(x)$ Pattern

Find $\int 4x^3 \sin x^4\, dx$

The integrands in Examples 1 and 2 fit the $f(g(x))g'(x)$ pattern exactly—you only had to recognize the pattern. You can extend this technique considerably with the Constant Multiple Rule

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

Many integrands contain the essential part (the variable part) of $g'(x)$ but are missing a constant multiple. In such cases, you can multiply and divide by the necessary constant multiple, as shown in Example 3.

Ex.3 Multiplying and Dividing by a Constant

Find $\int t^3\sqrt{t^4+5}\,dt$

Change of Variables

With a formal **change of variables,** you completely rewrite the integral in terms of u and du (or any other convenient variable). Although this procedure can involve more written steps than the pattern recognition illustrated in Examples 1 to 3, it is useful for complicated integrands. The change of variables technique uses the Leibniz notation for the differential. That is, if $u = g(x)$, then $du = g'(x)\,dx$, and the integral in Theorem 4.13 takes the form

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C.$$

Ex.4 Change of Variables

Find $\displaystyle\int \frac{x^2}{(16 - x^3)^2}\,dx$

Ex.5 Change of Variables

Find $\int x\sqrt{4x + 1}\, dx$

Ex.6 Change of Variables

Find $\int \sqrt{\tan x}\, \sec^2 x \, dx$

Guidelines for Making a Change of Variables

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)\,dx$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

The General Power Rule for Integration

One of the most common u-substitutions involves quantities in the integrand that are raised to a power. Because of the importance of this type of substitution, it is given a special name—the **General Power Rule for Integration.** A proof of this rule follows directly from the (simple) Power Rule for Integration, together with Theorem 4.13.

THEOREM 4.13 The General Power Rule for Integration

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C, \qquad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1.$$

Ex.7 Substitution and the General Power Rule

a. $\int 3(3x - 1)^4\,dx =$

Ex.7 Substitution and the General Power Rule

b. $\int (2x + 1)(x^2 + x)\, dx =$

c. $\int 3x^2 \sqrt{x^3 - 2}\, dx =$

d. $\int \frac{-4x}{(1 - 2x^2)^2}\, dx =$

e. $\int \cos^2 x \sin x\, dx =$

Change of Variables for Definite Integrals

When using u-substitution with a definite integral, it is often convenient to determine the limits of integration for the variable u rather than to convert the antiderivative back to the variable x and evaluate at the original limits. This change of variables is stated explicitly in the next theorem. The proof follows from Theorem 4.13 combined with the Fundamental Theorem of Calculus.

THEOREM 4.14 Change of Variables for Definite Integrals

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

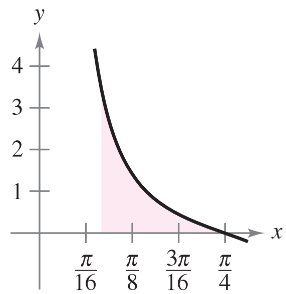
$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Ex.8 Change of Variables

Find $\displaystyle\int_0^2 \frac{x}{\sqrt{1+2x^2}}\,dx$

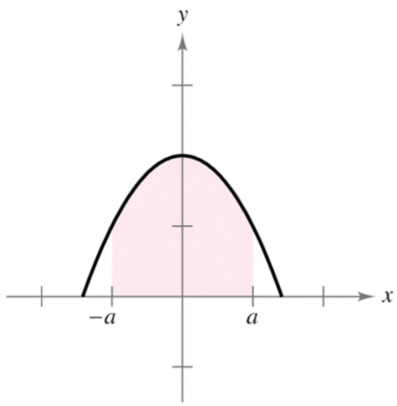
Ex.9 Change of Variables

$$\text{Find } \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x \, dx$$

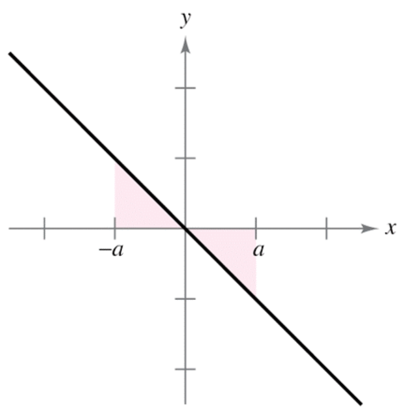


Integration of Even and Odd Functions

Even with a change of variables, integration can be difficult. Occasionally, you can simplify the evaluation of a definite integral over an interval that is symmetric about the y-axis or about the origin by recognizing the integrand to be an even or odd function (see Figure 4.40).



Even function



Odd function

Figure 4.40

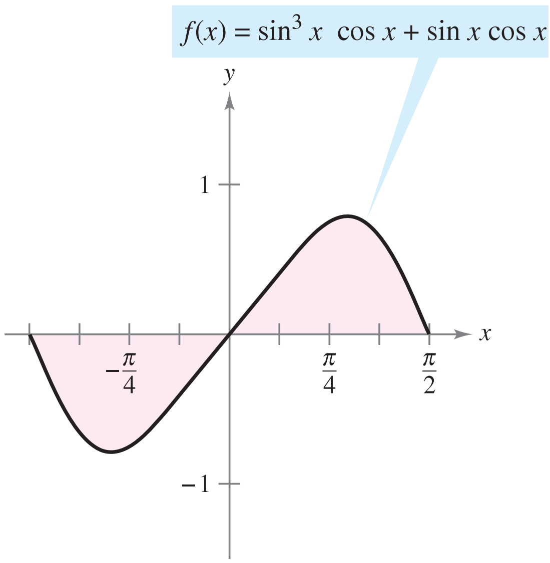
THEOREM 4.15 Integration of Even and Odd Functions

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an *even* function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
2. If f is an *odd* function, then $\int_{-a}^{a} f(x)\,dx = 0$.

Ex.10 Integration of and Odd Function

Evaluate $\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\,dx$.



Because f is an odd function,

$$\int_{-\pi/2}^{\pi/2} f(x)\, dx = 0.$$

Figure 4.41

NOTE From Figure 4.41 you can see that the two regions on either side of the y-axis have the same area. However, because one lies below the x-axis and one lies above it, integration produces a cancellation effect. (More will be said about this in Section 7.1.) ■